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SHOCK ADIABAT OF A ONE-VELOCITY HETEROGENEOUS MEDIUM

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An expression for the shock adiabat of a multicomponent mixture has been obtained. It agrees with the equations of the one-velocity model of a heterogeneous medium. The predictions are compared with the generally used shock adiabat equation based on the assumption on additive shock compression of mixture components.

The one-velocity model of a heterogeneous medium is widely used to simulate various problems in foamed and bubble liquids [1–3]. The shock adiabat of a multicomponent mixture in the multiphase hydrodynamics plays the same important role as the gas adiabat in the single-phase gas dynamics. It is used in constructing self-similar solutions and its expression enters as fragments of a code into the computer programs of a number of numerical methods.

In the literature concerned with the derivation of the shock adiabat equation of a mixture, the assumption is made that on shock compression each component of the mixture is compressed following its own shock adiabat [4]. It seems that the principle of the additive shock compressibility of fractions was first formulated in [5]. When it is applied to an n-component mixture in which the first m components are compressible, the shock adiabat equation takes the form

$$\frac{\rho_0}{\rho_s} = \sum_{i=1}^{m} \alpha_{i0} \left(\frac{\chi_i (p_s + p_{*i}) + p_0 + p_{*i}}{\chi_i (p_0 + p_{*i}) + p_s + p_{*i}} \right) + \sum_{j=m+1}^{n} \alpha_{j0} , \quad (1)$$

where $\chi_i = (\gamma_i - 1)/(\gamma_i + 1)$ and $p_{*i} = \rho_{*i} c_{*i}^2/\gamma_i$. Equation (1) correlates the values for the pressure and density of the mixture on different sides of the shock front, and it is valid for a mixture of fractions the behavior of which is described by the equation of state of the form

$$\varepsilon_i = \frac{p - c_{*i}^2 (\rho_i^0 - \rho_{*i})}{\rho_i^0 (\gamma_i - 1)} . \quad (2)$$

An individual shock adiabat corresponding to this equation is

$$\rho_{is}^0 = \rho_{i0}^0 \left(\frac{\chi_i (p_0 + p_{*i}) + p_s + p_{*i}}{\chi_i (p_s + p_{*i}) + p_0 + p_{*i}} \right). \quad (3)$$

The expression for the shock adiabat of the mixture (1) follows from the formula

$$\frac{\rho_0}{\rho_s} = \sum_{i=1}^{m} \frac{\alpha_{i0} \rho_{i0}^0}{\rho_{is}^0} + \sum_{j=m+1}^{n} \alpha_{j0} , \quad (4)$$

which is valid along the trajectory of the motion of a particle of the medium [4] (here relations (3) were taken into account).

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It should be noted that the question as to the validity of the assumption on the independence of shock compression of the mixture components at transition through the shock-wave front has not been discussed in the literature.

We will obtain the shock adiabat of the mixture without resorting to the hypothesis concerning the additive shock compression of the mixture components. We will avail ourselves of the integral equations of the mass, momentum, and energy conservation laws for both the mixture as a whole and its ith fraction, which, in the case of a one-dimensional plane flow in the phase plane (x, t), have the form

$$\oint_C (\rho dx - \rho u dt) = 0\,, \quad \oint_C (\rho u dx - (p + \rho u^2)\, dt) = 0\,,$$

$$\oint_C \left\{\rho\,(\varepsilon + u^2/2)\, dx - [\rho u\,(\varepsilon + u^2/2 + p/\rho)]\, dt\right\} = 0\,,$$

$$\oint_C (\rho_i dx - \rho_i u dt) = 0\,, \quad \oint_C (\rho_i u dx - (\alpha_i p + \rho_i u^2)\, dt) = \sum_{j=1}^{n_{(j\neq i)}} \iint_\Omega f_{ij} dx dt\,, \tag{5}$$

$$\oint_C \left\{\rho_i\,(\varepsilon_i + u^2/2)\, dx - [\rho_i u\,(\varepsilon_i + u^2/2 + p/\rho_i^0)]\, dt\right\} = \sum_{j=1}^{n_{(j\neq i)}} \iint_\Omega f_{ij} u dx dt\,.$$

Here, f_{ij} is the density of the force of the interfraction interaction between the ith and jth components of the mixture; $\rho_i = \rho_i^0\alpha_i$ is the reduced density of the mixture; $\rho = \sum_{i=1}^{n} \rho_i^0\alpha_i$. In Eqs. (5), the terms corresponding to external forces, energy sources, and heat fluxes are omitted. Note that the expression

$$\rho\varepsilon = \sum_{i=1}^{n} \rho_i\varepsilon_i \tag{6}$$

can be considered as the equation of state of the mixture as a whole. In the particular case of a binary mixture of an ideal gas (with adiabatic index γ) and an incompressible component, Eq. (6) yields

$$\varepsilon = \frac{\alpha p}{(\gamma - 1)\, p} + \frac{Z\,(1-\alpha)}{\rho}\,, \tag{7}$$

where $Z = \varepsilon_2\rho_2^0$ = const. But if a binary mixture with two compressible fractions is considered, each obeying the equation of state (2), then Eq. (6), subject to relations $\alpha_2 = 1 - \alpha_1$ and $\rho_2^0 = (\rho - \alpha_1\rho_1^0)/(1 - \alpha_1)$, yields an equation of state for the mixture as a whole:

$$\varepsilon = \frac{(B_{12}p - a_{12}\rho_1^0 + b_{12})\,\alpha_1 + pB_2 + b_2}{\rho} - a_2\,, \tag{8}$$

where $B_1 = 1/(\gamma_1 - 1)$, $B_2 = 1/(\gamma_2 - 1)$; $B_{12} = B_1 - B_2$; $a_1 = c_{*1}^2/(\gamma_1 - 1)$; $a_2 = c_{*2}^2/(\gamma_2 - 1)$; $a_{12} = a_1 - a_2$; $b_1 = a_1\rho_{*1}$; $b_2 = a_2\rho_{*2}$, and $b_{12} = b_1 - b_2$.

In considering an n-component mixture with the first m compressible fractions, each of which obeys state equation (2), Eq. (6) takes the form

$$\varepsilon = \frac{1}{\rho}\left\{\sum_{i=1}^{m-1} (pB_{im} - a_{im}\rho_i^0 + b_{im})\,\alpha_i + pB_m + b_m + \sum_{j=m+1}^{n} \alpha_j\rho_j^0\varepsilon_j\right\} - a_m\,, \tag{9}$$

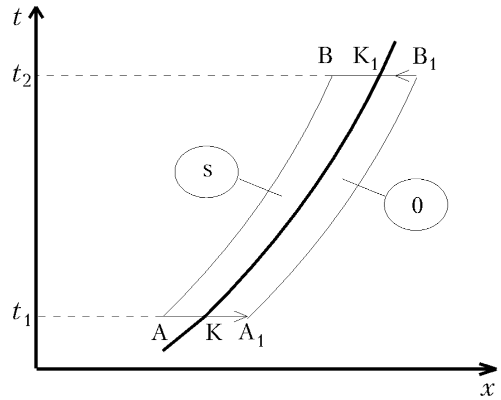

Fig. 1. Toward the derivation of relations on shock discontinuity.

where, by analogy with (8), the following coefficients are introduced: $B_i = 1/(\gamma_i - 1)$, $a_i = B_i c_{*i}^2$, $b_i = a_i \rho_{*i}$, $B_{im} = B_i - B_m$, $a_{im} = a_i - a_m$, and $b_{im} = b_i - b_m$. With account for these coefficients, the equation of the state of the ith fraction can be rewritten as

$$\varepsilon_i = (pB_i + b_i)/\rho_i^0 - a_i \,. \tag{10}$$

As the contour C entering into Eq. (5), we will select the contour A_1B_1BA (Fig. 1), where the KK_1 line is the trajectory of the shock front $x(t)$; therefore the relation $dx/dt = D(t)$ is valid along it. In the contour C considered, we will bring together the sides of the curvilinear tetrahedral so that they remain on each side of the discontinuity line $x(t)$. The integrals over the upper and lower bases AA_1 and BB_1, just as the integrals on the right sides of equalities (5), tend to zero. Here, for example, the first of equations (5) goes over in the limit into the relation

$$\int_{K_1}^{K} (\rho_s dx - \rho_s u_s dt) + \int_{K}^{K_1} (\rho_0 dx - \rho_0 u_0 dt) = 0 \,. \tag{11}$$

Taking into account the fact that along the discontinuity line KK_1 the expression $dx = Ddt$ is valid, Eq. (11) will be rewritten as

$$-\int_{t_1}^{t_2} (\rho_s D - \rho_s u_s)\, dt + \int_{t_1}^{t_2} (\rho_0 D - \rho_0 u_0)\, dt = 0 \,,$$

whence, by virtue of the arbitrariness of t_1 and t_2, it follows that

$$\rho_s\, (u_s - D) = \rho_0\, (u_0 - D) \,. \tag{12}$$

Proceeding in a similar way with the equations of system (5), we obtain

$$u_s \rho_s\, (u_s - D) + p_s = u_0 \rho_0\, (u_0 - D) + p_0 \,, \tag{13}$$

$$\rho_s\, (u_s - D)\, (\varepsilon_s + u_s^2/2) + p_s u_s = \rho_0\, (u_0 - D)\, (\varepsilon_0 + u_0^2/2) + p_0 u_0 \,, \tag{14}$$

$$\rho_{is}\, (u_s - D) = \rho_{i0}\, (u_0 - D) \,, \tag{15}$$

$$\rho_{is} u_s\, (u_s - D) + \alpha_{is} p_s = \rho_{i0} u_0\, (u_0 - D) + \alpha_{i0} p_0 \,, \tag{16}$$

$$\rho_{is}\,(u_s - D)\,(\varepsilon_{is} + u_s^2/2) + \alpha_{is} u_s p_s = \rho_{i0}\,(u_0 - D)\,(\varepsilon_{i0} + u_0^2/2) + \alpha_{i0} u_0 p_0\,. \tag{17}$$

Equations (12)–(17) represent the general form of the relations that couple the parameters on their side of the discontinuity surface. These relations generalize the well-known Rankine–Hugoniot relations to one-velocity multicomponent mixtures and express the laws of conservation of the mass-, momentum-, and energy fluxes through the discontinuity surface.

After a number of rearrangements, Eqs. (12)–(14) will yield the following relations:

$$D = \frac{\rho_s u_s - \rho_0 u_0}{\rho_s - \rho_0}\,, \tag{18}$$

$$u_s = u_0 + \sqrt{(p_s - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho_s}\right)}\,, \tag{19}$$

$$\varepsilon_s - \varepsilon_0 = \frac{p_s + p_0}{2}\left(\frac{1}{\rho_0} - \frac{1}{\rho_s}\right). \tag{20}$$

A comparison of (12) and (15) gives

$$\alpha_{is} = \alpha_{i0}\,\frac{\rho_s \rho_{i0}^0}{\rho_0 \rho_{is}^0}\,. \tag{21}$$

In the particular case of a binary mixture of an ideal gas (with adiabatic index γ) and an incompressible component, the equation of the law of conservation of a mass flux through the shock front for the incompressible fraction will take the form

$$(1 - \alpha_s)\,(u_s - D) = (1 - \alpha_0)\,(u_0 - D)\,. \tag{22}$$

After a number of transformations, Eqs. (7), (12), (20), and (22) yield the shock adiabat of the mixture:

$$\frac{\rho_0}{\rho_s} = 1 - \frac{2\alpha_0\,(p_s - p_0)}{p_s\,(\gamma - 1) + p_0\,(\gamma + 1)}\,,$$

which coincides exactly with the corresponding expression from Eq. (1).

In considering a binary mixture with two compressible fractions, we rewrite (17), using (10), (18), and (21):

$$\rho_{1s}^0 = \frac{p_s B_1 + b_1 - \dfrac{u_s\,p_s\,(\rho_s - \rho_0)}{\rho_0\,(u_s - u_0)}}{\dfrac{u_0^2}{2} - \dfrac{u_s^2}{2} + \dfrac{1}{\rho_{10}^0}\left(p_0 B_1 + b_1 - \dfrac{u_0 p_0\,(\rho_s - \rho_0)}{\rho_s\,(u_s - u_0)}\right)}\,. \tag{23}$$

Expression (20), subject to (8), will take the form

$$\frac{(p_s B_{12} - a_{12}\rho_{1s}^0 + b_{12})\,\alpha_{1s} + p_s B_2 + b_2}{\rho_s} - \frac{(p_0 B_{12} - a_{12}\rho_{10}^0 + b_{12})\,\alpha_{10} + p_0 B_2 + b_2}{\rho_0} = \frac{p_s + p_0}{2}\left(\frac{1}{\rho_0} - \frac{1}{\rho_s}\right). \tag{24}$$

Substituting, into Eq. (24), the expression for the volumetric fraction α_{1s} from (21) and ρ^0_{1s} from (23) and taking into account Eqs. (19), we obtain a relation which couples the pressure p_s and mixture density ρ_s behind the shock front precisely which is the adiabat of the mixture of two compressible fractions. Note that in the considered case of a binary mixture, relation (24) cannot already be reduced to the form of (1). This, in particular, is evident from the fact that the expression for the shock adiabat (24), in contrast to (1), depends also on the values of u_0 and ρ^0_{10}.

Similarly, we obtain an expression for the shock adiabat in the case of an n-component mixture with m compressible fractions:

$$\frac{1}{\rho_s}\left[\sum_{i=1}^{m-1}(p_s B_{im} - a_{im}\rho^0_{is} + b_{im})\,\alpha_{is} + p_s B_m + b_m + \sum_{j=m+1}^{n}\alpha_{js}\rho^0_j\varepsilon_j\right] -$$

$$-\frac{1}{\rho_0}\left[\sum_{i=1}^{m-1}(p_0 B_{im} - a_{im}\rho^0_{i0} + b_{im})\,\alpha_{i0} + p_0 B_m + b_m + \sum_{j=m+1}^{n}\alpha_{j0}\rho^0_j\varepsilon_j\right] = \frac{p_s + p_0}{2}\left(\frac{1}{\rho_0} - \frac{1}{\rho_s}\right),$$

which should be considered jointly with the relations

$$\rho^0_{is} = \frac{p_s B_i + b_i - \dfrac{u_s p_s(\rho_s - \rho_0)}{\rho_0(u_s - u_0)}}{\dfrac{u_0^2}{2} - \dfrac{u_s^2}{2} + \dfrac{1}{\rho^0_{i0}}\left[p_0 B_i + b_i - \dfrac{u_0 p_0(\rho_s - \rho_0)}{\rho_s(u_s - u_0)}\right]}, \quad i = 1, ..., m-1\,;$$

$$\alpha_{js} = \alpha_{j0}\frac{\rho_s}{\rho_0}, \quad j = m+1, ..., n\,,$$

as well as with (19) and (21).

As an example of application of the shock adiabat obtained in this work, the problem of motion of a shock wave in a bubble liquid is considered. We assume that the shock wave moves over an immovable (u_0 = 0, $p_0 = 10^5$ Pa, α_{10} = 0.98) water–air mixture. The coefficients of the equation of state (2) for water are equal to: γ_1 = 5.59, ρ_{*1} = 1000 kg/m^3, c_{*1} = 1515 m/sec, and for the gas component to: γ_2 = 1.4, ρ_{*2} = 1.24 kg/m^3, and c_{*2} = 0.

Figure 2a presents the calculated dependences of the mixture density ρ_s (curves 1) and the physical density of the gas ρ^0_{1s} (2) behind the shock-wave front on dimensionless pressure p_s/p_0. The corresponding dependences for the velocity of the shock-compressed mixture $u_s(p_s/p_0)$ (curves 1) and the velocity of the shock-wave front motion $D(p_s/p_0)$ (2) are given in Fig. 2b. Figure 2c demonstrates the dependences of the volumetric fraction of the gas α_{1s} behind the shock front on the pressure ratio p_s/p_0. The dashed and solid curves in Fig. 2 present the data obtained in calculations performed with the use of shock adiabats (1) and (24).

The closeness of the values of the mixture velocity as well as of the mixture density behind the shock front that were calculated with the aid of shock adiabats (1) and (24) is evident. However, the shock-wave velocity calculated with the use of shock adiabat (1) appears to be somewhat higher than that calculated with the use of (24).

But the fundamental difference between the results based on approximation of shock adiabats (1) and (24) is that, in the case of using the shock adiabat coordinated with the equations of motion, the degree of compression of the gas component turns out to be substantially higher (Fig. 2a) than in the case of using shock adiabat (1) It is known from the gas dynamics that the maximum degree of compression of an ideal gas in a shock wave is defined by the ratio $(\gamma+1)/(\gamma-1)$. At γ = 1.4, the gas cannot be compressed by a shock wave by more than sixfold. But in a mixture this value may be higher (see Fig. 2a). It should also be noted that in using a refined shock adiabat, the calculated volumetric fraction of the gas in the mixture is substantially lower than for (1). The results of computations turn out to be close to those given by the isothermal model of a bubble liquid, which is often used to explain a number of experimental effects [6]. It should be noted that in the isothermal model for the shock wave moving over the

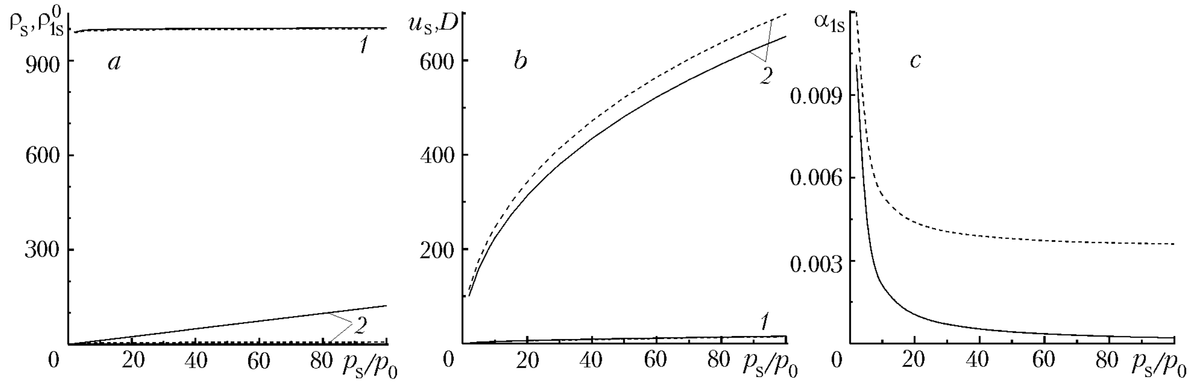

Fig. 2. Dependences of the flow parameters ρ_s, ρ^0_{1s}, u_s, D, and α_{1s} behind the shock-wave front on the shock-wave intensity p_s/p_0.

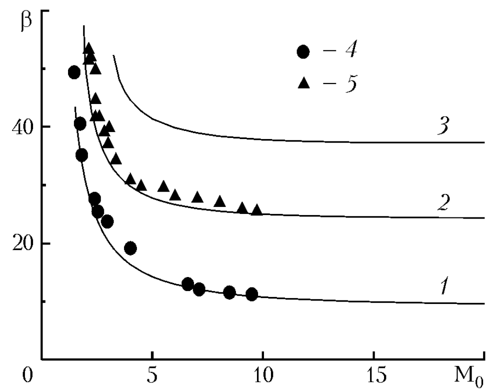

Fig. 3. The angle β vs. the Mach number M_0 in the stream incident on the wedge.

bubble liquid (with an incompressible liquid fraction), the gas concentration behind its front is equal to zero because all gas bubbles collapse entirely.

As another example of application of the shock adiabat, we will calculate the flow of a water–air mixture (α_{10} = 0.56) near a wedge for the mode with an attached shock wave. This problem was experimentally investigated in [7]. The corresponding calculation dependences were obtained for the wedge angles ϑ = 4, 10, and 14^o (Fig. 3, curves 1–3) from the system of equations which express the kinematic and dynamic conditions of mixture motion near the wedge and which are considered jointly with the expression for the shock adiabat of the mixture:

$$\rho_s = \rho_0 \frac{\tan\beta}{\tan(\beta-\vartheta)}, \quad u_s = u_0 \frac{\cos\beta}{\cos(\beta-\vartheta)}, \quad p_s = p_0 + \rho_0 u_0^2 \sin^2\beta \left[1 - \frac{\tan(\beta-\vartheta)}{\tan\beta}\right],$$

$$\frac{1}{\rho_s}\left[(p_s B_{12} - a_{12}\rho^0_{1s} + b_{12})\,\alpha_{1s} + p_s B_2 + b_2\right] - \frac{1}{\rho_0}\left[(p_0 B_{12} - a_{12}\rho^0_{10} + b_{12})\,\alpha_{10} + p_0 B_2 + b_2\right] = \frac{p_0 + p_s}{2}\left(\frac{1}{\rho_0} - \frac{1}{\rho_s}\right),$$

$$\rho^0_{1s} = \frac{p_s B_1 + b_1 - \dfrac{p_s u_s(\rho_s - \rho_0)}{\rho_0 (u_s - u_0)}}{\dfrac{u_0^2}{2} - \dfrac{u_s^2}{2} + \dfrac{1}{\rho^0_{10}}\left[p_0 B_1 + b_1 - \dfrac{p_0 u_0 (\rho_s - \rho_0)}{\rho_s (u_s - u_0)}\right]}, \tag{25}$$

$$\alpha_{1s} = \frac{\alpha_{10}\rho_s\rho_{10}^0\left\{\frac{u_0^2}{2} - \frac{u_s^2}{2} + \frac{1}{\rho_{10}^0}\left[p_0B_1 + b_1 - \frac{p_0\,u_0\,(\rho_s - \rho_0)}{\rho_s\,(u_s - u_0)}\right]\right\}}{\rho_0\left[p_sB_1 + b_1 - \frac{p_s\,u_s\,(\rho_s - \rho_0)}{\rho_0\,(u_s - u_0)}\right]} .$$

Calculations by formulas (25), in which the expression for the shock adiabat, coordinated with the equation of mixture motion, is used, give results close to the experimental values from [7] (Fig. 3, dependences 4 and 5 for $\vartheta = 4$ and 10^o). Here, the assumption on the isothermicity of the process of shock compression of the mixture as well as the condition of incompressibility of the liquid fraction adopted in [7] are not used. It should also be noted that the application of the condition on the isothermicity of the process excludes from consideration the effect of sonoluminescence (luminescence) observed in experiments on shock compression in bubble media.

NOTATION

C, contour bounding the region of integration Ω; c_{*i}, constant of the equation of state, m/sec; D, velocity of motion of the shock front, m/sec; M, Mach number; m, number of compressible fractions in a mixture; n, total number of fractions in a mixture; p, pressure, Pa; t, time, sec; u, velocity, m/sec; x, space coordinate, m; α_i, volumetric fraction of the ith component of the mixture; β, angle of inclination of the attached shock wave to the horizontal, deg; γ_i, a constant of the equation of state; ε, specific internal energy, m^2/sec^2; ϑ, angle of inclination of the wedge plane to the horizontal, deg; ρ, mixture density, kg/m^3; ρ_i^0, true density of the ith individual component, kg/m^3; ρ_{*i}, a constant of the equation of state, kg/m^3. Superscripts and subscripts: 0, initial state; s, behind the shock wave front.

REFERENCES

1. V. S. Surov, Interaction of shock waves with droplets of a bubble liquid, *Zh. Tekh. Fiz.*, **71**, No. 6, 17–22 (2001).
2. V. S. Surov, One-velocity heterogeneous flow near a cone, *Inzh.-Fiz. Zh.*, **75**, No. 2, 48–52 (2002).
3. V. S. Surov, Modeling of the impact of a bubble-liquid droplet, *Inzh.-Fiz. Zh.*, **75**, No. 4, 135–140 (2002).
4. Kh. A. Rakhmatulin, Propagation of waves in multicomponent media, *Prikl. Mat. Mekh.*, **33**, No. 4, 598–601 (1969).
5. A. N. Dremin and I. A. Karpukhin, Method for determining shock adiabats for disperse systems, *Prikl. Mekh. Tekh. Fiz.*, No. 3, 184–188 (1960).
6. V. S. Surov, Calculation of shock-wave processes in bubble liquids, *Zh. Tekh. Fiz.*, **68**, No. 11, 12–19 (1998).
7. R. B. Eddington, Investigation of supersonic phenomena in a two-phase (liquid–gas) tunnel, *AIAA J.*, **8**, No. 1, 65–74 (1970).